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Lattice Kinetic Formulation for Ferrofluids

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The lattice Boltzmann approach is used to solve continuum equations describing colloids of ferromagnetic particles (ferrofluids) in a regime, where the particle spins are in equilibrium with magnetic torques. This limit of rapid spin adjustment yields a symmetric total stress tensor that is essential for a kinetic formulation based on the Boltzmann equation. The magnetisation equation is solved using a vector-valued distribution function analogous to the earlier treatment (*J. Comput. Phys.* **179**, 95) of the induction equation in magnetohydrodynamics, but the details are rather more complex because the magnetisation equation is not in conservation form except in a weakly magnetised limit.

KEY WORDS: Complex fluids; ferromagnetic liquids; lattice Boltzmann equations; magnetoviscosity; multiple relaxation time collision operators; polar fluids.

1. INTRODUCTION

Ferrofluids are colloids of tiny (10 nm) single-domain ferromagnetic particles suspended in an insulating liquid such as toluene.[(1−3)]. First synthesised in 1964, they have evolved from a laboratory curiosity to important technological materials, with applications such as high-performance seals and bearings. Ferrofluids also raise interesting questions in basic fluid mechanics, since a continuum description of a ferrofluid sometimes requires an asymmetric stress tensor, or couple stress. This arises from the torque $\mathbf{M}\times\mathbf{B}$ exerted when the induced magnetisation $\mathbf{M}$ of the ferromagnetic particles is inclined to the magnetic induction $\mathbf{B}$. The usual argument for symmetry of the stress tensor is incorrect for colloids. A volume of lengthscale L contains $O(L^3)$ suspended particles that are free to rotate relative

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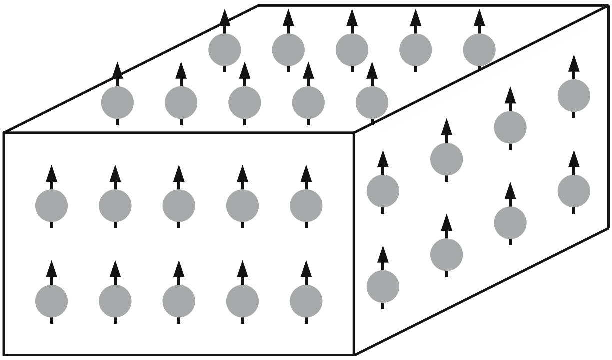

Fig. 1. Suspended particles with individual spins in a ferrofluid.

to the fluid under magnetic torques, as sketched in Fig. 1, so its moment of inertia scales as L^3, rather than as L^5 as usually assumed. However, this paper considers the rapid spin relaxation limit, for which the asymmetric viscous and Maxwell stresses combine into a symmetric total stress.

The study of ferrofluids differs from magnetohydrodynamics (MHD) that concerns itself with nonmagnetisable but electrically conducting fluids, so the key ingredient is the Lorentz force $\mathbf{J}\times\mathbf{B}$ generated by flowing currents $\mathbf{J}$. While the magnetic manipulation of liquid metals, say, by the Lorentz force is usually impractical due to resistive losses, these losses are absent in insulating ferrofluids, where $\mathbf{J}=0$. Their rheology is thus readily manipulable by weak (10 mT) magnetic fields.

Lattice Boltzmann equations are becoming a very popular simulation tool in fluid dynamics,[4,5] but have attracted less attention in MHD. The lattice Boltzmann approach expresses macroscopic quantities like fluid density, velocity, or stress, as moments of a distribution function. Macroscopic evolution equations arise in turn from moments of the postulated evolution equation for the distribution function, and naturally take the form of the density evolving through the divergence of the momentum, the momentum evolving through the divergence of a stress, and so on. Thus it is straightforward to incorporate the Lorentz force due to a magnetic field by adjusting the distribution function to obtain the magnetic contribution to the total stress.[6–8] Moreover, the viscous stress is also available from the distribution function, and this may be manipulated to simulate non-Newtonian fluids with shear-dependent viscosities,[9] and even some kinds of viscoelastic behavior such as a Jeffreys viscoelastic fluid.[10]

However, the usual scalar distribution function borrowed from the kinetic theory of gases cannot describe evolution equations like

$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$ (see Section 2). An early formulation[6,7] of MHD used a tensor-valued distribution function for both fluid and magnetic quantities, and was subsequently modified to simulate ferrofluids in a weak magnetisation limit.[11] This paper explores more general ferrofluid equations based on the author's reformulation of MHD[8] using a separate vector-valued distribution function for **B**. In fact, a vector distribution function had already been used[12] to evolve the magnetisation in simulations of nuclear magnetic resonance (NMR) in normal fluids like water, and various vector and tensor distributions have been used in continuum models of liquid crystals.[13,14]

We follow standard ferrofluid conventions,[1] which differ somewhat from conventional fluid dynamics and MHD. The symbol $\boldsymbol{\omega}$ is used for intrinsic angular velocity or spin, which at equilibrium is *half* the usual fluid vorticity, $\boldsymbol{\omega} \approx \frac{1}{2}\nabla \times \mathbf{u}$. In this context "spin" means rotation of the suspended particles. It is also conventional to use Gaussian units in which the magnetisation (6) retains a factor of 4π.

2. LATTICE BOLTZMANN APPROACH TO HYDRODYNAMICS

The lattice Boltzmann approach to hydrodynamics expresses macroscopic quantities like density ρ, velocity $\mathbf{u}$, and momentum flux $\mathbf{\Pi}$ as moments of an underlying distribution function $f_i(\mathbf{x}, t)$:

$$\rho = \sum_{i=0}^{N} f_i, \quad \rho \mathbf{u} = \sum_{i=0}^{N} \boldsymbol{\xi}_i f_i, \quad \mathbf{\Pi} = \sum_{i=0}^{N} \boldsymbol{\xi}_i \boldsymbol{\xi}_i f_i, \tag{1}$$

where the constant vectors $\boldsymbol{\xi}_i$ correspond to particle velocities in kinetic theory. Postulating evolution of the f_i by the discrete Boltzmann–BGK equation

$$\partial_t f_i + \boldsymbol{\xi}_i \cdot \nabla f_i = -\frac{1}{\tau_s}\left(f_i - f_i^{(0)}\right), \tag{2}$$

then implies mass and momentum conservation equations as moments of (2):

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \partial_t (\rho \mathbf{u}) + \nabla \cdot \mathbf{\Pi} = 0, \tag{3}$$

provided the equilibria $f_i^{(0)}$ are chosen so that $\sum_{i=0}^{N} f_i^{(0)} = \rho$ and $\sum_{i=0}^{N} \boldsymbol{\xi}_i f_i^{(0)} = \rho \mathbf{u}$. For suitable choices of the $\boldsymbol{\xi}_i$ and $f_i^{(0)}$, the implied conservation equations (3) may be shown to coincide with the compressible

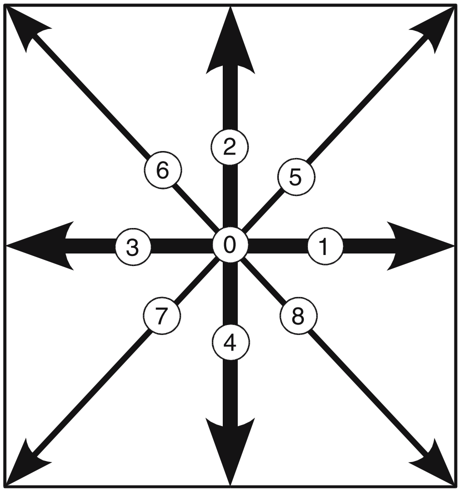

Fig. 2. Two-dimensional nine velocity (D2Q9) lattice for the fluid, and five velocity subset (D2Q5) for the magnetic field.

Navier–Stokes equations in the slowly varying (small τ_s) limit. In two dimensions, the $\boldsymbol{\xi}_i$ are most commonly chosen to form a square lattice as shown in Fig. 2.

The difficulty with extending this approach to magnetohydrodynamics, or ferrofluids, is that the momentum flux tensor $\mathbf{\Pi}$ appearing in (1) and (3) is symmetric by definition. By contrast, the magnetic field $\mathbf{B}$ evolves according to

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \text{or} \quad \partial_t \mathbf{B} + \nabla \cdot \mathbf{\Lambda} = 0, \tag{4}$$

where the flux $\mathbf{\Lambda}$ is an *antisymmetric* tensor with components $\Lambda_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} E_\gamma$. It is thus impossible to obtain the correct induction equation (4) from a moment of a scalar equation like (2). Rather than use a tensor-valued distribution function for both fluid and magnetic variables,[6,7] Dellar[8] introduced a separate vector-valued distribution function $\mathbf{g}_i$ such that

$$\mathbf{B} = \sum_{i=0}^{N} \mathbf{g}_i, \quad \mathbf{\Lambda} = \sum_{i=0}^{N} \boldsymbol{\xi}_i \mathbf{g}_i, \quad \partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau_B}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right). \tag{5}$$

This formulation was shown to be effective at simulating resistive MHD, for which $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \nabla \times \mathbf{B}$, with resistivity $\eta \propto \tau_B$. Although associating a vector with each lattice direction requires two or three times more storage than a scalar, it is possible to use fewer lattice directions without destroying isotropy.[8] Five lattice vectors suffice in two dimensions, as shown by the thick lines in Fig. 2, whereas nine lattice vectors for the hydrodynamic distribution functions f_i are necessary for an isotropic viscous stress.

3. CONTINUUM DESCRIPTION OF FERROFLUIDS

In ferrofluids, it is important to distinguish between the magnetic induction $\mathbf{B}$ and the magnetic field $\mathbf{H}$. In a vacuum, the two are simply related by $\mathbf{B} = \mu_0\mathbf{H}$, where the constant of proportionality μ_0 is called the permeability of free space. However, this simple relation is typically lost when one averages the vacuum Maxwell equations, with their associated microscopic charges and currents at the atomic level, to derive macroscopic or effective Maxwell equations for continuous media.[15,16] The macroscopic quantity normally written $\mathbf{B}$ is the average of the microscopic magnetic field, and satisfies $\nabla\cdot\mathbf{B} = 0$ because averaging commutes with differentiation. To account for the average of the microscopic currents in the inhomogeneous Maxwell equations, one typically relates $\mathbf{H}$ to $\mathbf{B}$ via

$$\mathbf{B} = \mathbf{H} + 4\pi\,\mathbf{M}, \tag{6}$$

in the Gaussian units commonly used in the ferrofluids literature. The vector $\mathbf{M}$ is the magnetisation, for which $\nabla\times\mathbf{M}$ is the effective current density that arises from averaging the microscopic currents due to moving charges. One may also interpret $\mathbf{M}$ as the average magnetic moment per unit volume in the continuum, which has the advantage of eliminating the gauge uncertainty arising from equating only $\nabla\times\mathbf{M}$ with a physical quantity.

The macroscopic Maxwell equations for an insulating material are then

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{H} = 0. \tag{7}$$

We neglect Maxwell's displacement current, as in magnetohydrodynamics, since it is tiny for materials moving nonrelativistically. The macroscopic current is then just $\nabla\times\mathbf{H}$, which vanishes in insulating materials. The magnetisation is negligible in the media usually treated in magnetohydrodynamics, so one typically just identifies $\mathbf{B}$ and $\mathbf{H}$, and often writes the current as $\nabla\times\mathbf{B}$ instead of $\nabla\times\mathbf{H}$.

The magnetisation $\mathbf{M}$ evolves according to the equation

$$\partial_t \mathbf{M} + \mathbf{u} \cdot \nabla \mathbf{M} = \boldsymbol{\omega} \times \mathbf{M} - \frac{1}{\tau_{\mathrm{M}}} (\mathbf{M} - \mathbf{M}_0), \tag{8}$$

where $\mathbf{u}$ and $\boldsymbol{\omega}$ are the fluid velocity and spin. This formula holds for incompressible fluids, where $\nabla \cdot \mathbf{u} = 0$. The equilibrium magnetisation $\mathbf{M}_0$ is usually modelled by the Langevin formula[1,3]

$$\mathbf{M}_0 = nmL(\xi)\hat{\mathbf{H}}, \quad \xi = m|\mathbf{H}|/k_{\mathrm{B}}T, \quad L(\xi) = \coth \xi - 1/\xi, \tag{9}$$

where $\hat{\mathbf{H}}$ is a unit vector parallel to $\mathbf{H}$, m is the magnetic moment of a single ferromagnetic particle, and n the number density of particles. The temperature is T, and k_{B} is Boltzmann's constant, so the parameter ξ represents the ratio of the energy due to dipoles interacting with the magnetic field to the thermal fluctuations tending to randomize dipole orientations. Relaxation of the magnetisation with timescale τ_{M} is due to a combination of Brownian motion and the Néel effect, i.e., physical rotation of the ferromagnetic particles, and rotation of the magnetic dipole moments relative to stationary particles. The relative importance of these two effects depends on the size of the particles.

The momentum flux may be written as $\boldsymbol{\Pi} = \rho \mathbf{u}\mathbf{u} - \boldsymbol{\sigma}$, defining the deviatoric stress $\boldsymbol{\sigma}$. For polar materials like ferrofluids the usual linear momentum equation

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot (\boldsymbol{\sigma}^{\mathrm{hydro}} + \boldsymbol{\sigma}^{\mathrm{mag}}), \tag{10}$$

equivalent to (3), must be supplemented by a second equation for the spin, or internal angular momentum[1,3]

$$\rho I \frac{D\boldsymbol{\omega}}{Dt} = 2\zeta (\nabla \times \mathbf{u} - 2\boldsymbol{\omega}) + \mathbf{M} \times \mathbf{H}. \tag{11}$$

Here I represents the moments of inertia of all the ferromagnetic particles in a unit volume. The spin viscosity ζ represents the viscous drag on particles rotating relative to the surrounding fluid. The corresponding viscous stress tensor for an incompressible fluid in our notation is[17]

$$\sigma^{\mathrm{hydro}}_{\alpha\beta} = -p\delta_{\alpha\beta} + \mu_{\mathrm{s}} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right) + \zeta \epsilon_{\alpha\beta\gamma} [\nabla \times \mathbf{u} - 2\boldsymbol{\omega}]_\gamma, \tag{12}$$

where μ_s is the usual shear viscosity, and the extra antisymmetric term is due to spin viscosity. Equivalent formulas appeared previously in a different notation.[3] The Maxwell stress exerted by the magnetic field may be written as[3,18]

$$\boldsymbol{\sigma}^{\mathrm{mag}} = \frac{1}{4\pi}\mathbf{HH} + \mathbf{MH} - p_{\mathrm{m}}\mathsf{I}, \tag{13}$$

with magnetic pressure $p_{\mathrm{m}} = H^2/8\pi - 2\pi M^2$, after using (6) to eliminate $\mathbf{B}$.

While the equilibrium magnetisation $\mathbf{M}_0$ given by (9) is parallel to $\mathbf{H}$, the instantaneous magnetisation $\mathbf{M}$ need not be parallel to $\mathbf{H}$. Thus the magnetic stress (13) is generally asymmetric, like the spin viscous stress in (12). This gives rise to the couple force $\mathbf{M}\times\mathbf{H}$ in the spin equation (11).

However, the timescale for $\boldsymbol{\omega}$ to adjust to equilibrium under (11) is usually extremely short, 10^{-11} s according to [3], so we may replace (11) by the equilibrium approximation

$$\boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{u} + \frac{1}{4\zeta}\mathbf{M}\times\mathbf{H}. \tag{14}$$

The total stress $\boldsymbol{\sigma}^{\mathrm{hydro}} + \boldsymbol{\sigma}^{\mathrm{mag}}$ then becomes symmetric,[3,17] which is crucial for a lattice Boltzmann formulation of the momentum equation in the form (3), as explained above. Combining the asymmetric terms from (12) and (13), and using (14) to eliminate $\boldsymbol{\omega}$, we find

$$\begin{aligned}[\mathbf{MH}]_{\alpha\beta} + \zeta\epsilon_{\alpha\beta\gamma}[\nabla\times\mathbf{u} - 2\boldsymbol{\omega}]_\gamma &= M_\alpha H_\beta - \frac{1}{2}\zeta\epsilon_{\alpha\beta\gamma}[\mathbf{M}\times\mathbf{H}]_\gamma \\ &= \frac{1}{2}(M_\alpha H_\beta + H_\alpha M_\beta),\end{aligned} \tag{15}$$

so the total stress becomes a symmetric tensor [3],

$$\begin{aligned}\sigma_{\alpha\beta} = &-(p + p_{\mathrm{m}})\delta_{\alpha\beta} + \frac{1}{4\pi}H_\alpha H_\beta + \frac{1}{2}(M_\alpha H_\beta + H_\alpha M_\beta) \\ &+\mu_s\big(\partial_\alpha u_\beta + \partial_\beta u_\alpha\big).\end{aligned} \tag{16}$$

4. HYDRODYNAMIC EQUILIBRIUM DISTRIBUTIONS

The usual nine velocity fluid equilibria[19] may be determined by projecting the desired moments onto tensor Hermite polynomials.[20] The relevant expression, for the momentum flux $\theta\rho\mathsf{I}+\rho\mathbf{u}\mathbf{u}$, is

$$f_i^{(0)} = w_i\left(\rho + \frac{1}{\theta}\boldsymbol{\xi}_i\cdot(\rho\mathbf{u}) + \frac{1}{2\theta^2}(\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \theta\mathsf{I}):(\rho\mathbf{u}\mathbf{u})\right), \tag{17}$$

where the weights w_i for the D2Q9 lattice shown in Fig. 2 are $w_0=4/9$, $w_{1,2,3,4}=1/9$, and $w_{5,6,7,8}=1/36$.

This approach gives formulas[8] for the necessary terms to change the equilibrium momentum flux $\mathbf{\Pi}^{(0)}$ by $\Delta\mathbf{\Pi}$:

$$\Delta f_i^{(0)} = w_i\frac{1}{2\theta^2}(\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \theta\mathsf{I}):(\Delta\mathbf{\Pi} - \theta\mathsf{I}(\mathrm{Tr}\Delta\mathbf{\Pi})). \tag{18}$$

Recalling that $\mathbf{\Pi}=\rho\mathbf{u}\mathbf{u}-\boldsymbol{\sigma}$, we obtain

$$\Delta f_i^{(0)} = -w_i\frac{1}{2\theta^2}\Big((\boldsymbol{\xi}_i\cdot\mathbf{M})(\boldsymbol{\xi}_i\cdot\mathbf{H}) + \theta(2\theta-1-|\boldsymbol{\xi}_i|^2)\,\mathbf{M}\cdot\mathbf{H}\Big), \tag{19}$$

from the $\Delta\mathbf{\Pi}=-\frac{1}{2}(\mathbf{M}\mathbf{H}+\mathbf{H}\mathbf{M})$ part of (minus) the Maxwell stress. This formula holds in two dimensions, where $\mathrm{Tr}\,\mathsf{I}=2$. From the magnetic pressure term we obtain

$$\Delta f_i^{(0)} = p_{\mathrm{m}} w_i\frac{1}{2\theta^2}(1-2\theta)(|\boldsymbol{\xi}_i|^2-2\theta), \tag{20}$$

again in two dimensions. These formulas are not unique, because the formula (18) involves only six of the nine possible degrees of freedom in the nine equilibrium distribution functions at each lattice point. However, these formulas worked well for magnetohydrodynamics.

Notice that the desired change $\Delta\mathbf{\Pi}$ in the momentum flux is contracted with the symmetric tensor $(\boldsymbol{\xi}_i\boldsymbol{\xi}_i-\theta\mathsf{I})$ in the formula (18) for the change in the distribution function. Thus only the symmetric part of $\Delta\mathbf{\Pi}$ can contribute to the distribution function. This is consistent with $\mathbf{\Pi}^{(0)}=\sum_{i=0}^{N}\boldsymbol{\xi}_i\boldsymbol{\xi}_i f_i^{(0)}$ being a symmetric tensor by construction.

Thus it is impossible to incorporate any asymmetric component of the stress tensor into the equilibrium distribution in this way. Instead, one must include the divergence of the antisymmetric part as an explicit body force in the momentum equation. The body force will typically involve spatial derivatives of the field variables, and in general these must be calculated by some finite difference or other approximation, instead of arising naturally through the streaming term $\boldsymbol{\xi}_i \cdot \nabla f_i$ in the Boltzmann–BGK equation (2). This approach has been used to approximate continuum equations with asymmetric stresses that describe liquid crystals in various regimes.[13,14]

5. LATTICE KINETIC MAGNETISATION EQUATION

Using the equilibrium approximation (14) for $\boldsymbol{\omega}$, the magnetisation equation (8) becomes

$$\partial_t \mathbf{M} + \mathbf{u} \cdot \nabla \mathbf{M} = \frac{1}{2}(\nabla \times \mathbf{u}) \times \mathbf{M} + \frac{1}{4\zeta}(\mathbf{M} \times \mathbf{H}) \times \mathbf{M} - \frac{1}{\tau_{\mathrm{M}}}(\mathbf{M} - \mathbf{M}^{(0)}), \quad (21)$$

with $\mathbf{M}^{(0)} = \chi(|\mathbf{H}|)\mathbf{H}$ from (9). This equation is not in conservation form, but one may eliminate derivatives of $\mathbf{u}$ in favor of derivatives of $\mathbf{M}$:

$$\begin{aligned} \partial_t \mathbf{M} + \nabla \cdot \left(\mathbf{u}\mathbf{M} - \frac{1}{2}\mathbf{M}\mathbf{u} + \frac{1}{2}(\mathbf{u} \cdot \mathbf{M})\, \mathsf{I} \right) &= \frac{1}{2} u_k \nabla M_k - \frac{1}{2}\mathbf{u} \nabla \cdot \mathbf{M} \\ &\quad + \frac{1}{4\zeta}(\mathbf{M} \times \mathbf{H}) \times \mathbf{M} - \frac{1}{\tau_{\mathrm{M}}}(\mathbf{M} - \mathbf{M}^{(0)}), \end{aligned} \quad (22)$$

where $u_k \nabla M_k$ is the vector with ith component $u_k \partial_i M_k$, or symbolically as

$$\partial_t \mathbf{M} + \nabla \cdot \boldsymbol{\Lambda}^{(0)} = -\frac{1}{\tau_{\mathrm{M}}}(\mathbf{M} - \mathbf{M}^{(0)}) + \mathbf{S}, \quad (23)$$

where $\mathbf{S}$ denotes the remaining source terms on the right-hand side of equation (22).

By analogy with MHD[8] we postulate a vector distribution function $\mathbf{g}_i$,

$$\mathbf{M} = \sum_{i=0}^{N} \mathbf{g}_i, \quad \boldsymbol{\Lambda} = \sum_{i=0}^{N} \boldsymbol{\xi}_i \mathbf{g}_i, \quad \mathbf{g}_i^{(0)} = w_i \left(\mathbf{M} + \frac{1}{\theta} \boldsymbol{\xi}_i \cdot \boldsymbol{\Lambda}^{(0)} \right). \quad (24)$$

The usual lattice Boltzmann relaxation time controls diffusion of $\mathbf{M}$. This effect should be very small, much smaller than the diffusivity implied by τ_{M}, which suggests using a multiple relaxation time (MRT) collision operator[21]

$$\partial_t \mathbf{g}_i + \boldsymbol{\xi}_i \cdot \nabla \mathbf{g}_i = -\frac{1}{\tau_{\mathrm{D}}}\left(\mathbf{g}_i - \mathbf{g}_i^{(0)}\right) - \frac{1}{\tau_{\mathrm{M}}} w_i \left(\mathbf{M} - \mathbf{M}^{(0)}\right) + w_i \mathbf{S}. \tag{25}$$

From the zeroth and first moments we get (at leading order)

$$\partial_t \mathbf{M} + \nabla\cdot\boldsymbol{\Lambda}^{(0)} = -\frac{1}{\tau_{\mathrm{M}}}\left(\mathbf{M} - \mathbf{M}^{(0)}\right) + \mathbf{S}, \tag{26}$$

$$\partial_t \boldsymbol{\Lambda}^{(0)} + \nabla\cdot\left(\sum_i \boldsymbol{\xi}_i \boldsymbol{\xi}_i \mathbf{g}_i^{(0)}\right) = -\frac{1}{\tau_{\mathrm{D}}}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^{(0)}\right), \tag{27}$$

while the gradient of $\mathbf{M}$ is available from the non-equilibrium part of the magnetisation distribution function to evaluate $\mathbf{S}$,

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{(0)} - \theta \tau_{\mathrm{D}} \nabla \mathbf{M} + \cdots . \tag{28}$$

The rearrangement (22) is required because the vorticity $\nabla\times\mathbf{u}$ is not available from $\boldsymbol{\Pi} - \boldsymbol{\Pi}^{(0)}$, only the symmetric strain field $\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathsf{T}}$.

Constructing a fully discrete form of (25) is complicated by $\mathbf{M}$ not being an invariant of the full collision operator, but only of the first term $(\mathbf{g}_i^{(0)} - \mathbf{g}_i)/\tau_{\mathrm{D}}$. Integrating (25) along a characteristic for time Δt gives

$$\mathbf{g}_i(\mathbf{x} + \boldsymbol{\xi}_i \Delta t, t + \Delta t) - \mathbf{g}_i(\mathbf{x}, t) = \int_0^{\Delta t} \mathbf{R}_i(\mathbf{x} + \boldsymbol{\xi}_i s, t + s) ds, \tag{29}$$

where the right-hand side contains the source and relaxation terms from (25). Approximating this integral with the trapezium rule gives

$$\begin{aligned}\mathbf{g}_i(\mathbf{x}+\boldsymbol{\xi}_i \Delta t, t+\Delta t) - \mathbf{g}_i(\mathbf{x}, t) = {} & \frac{\Delta t}{2}\left(\mathbf{R}_i(\mathbf{x}+\boldsymbol{\xi}_i \Delta t, t+\Delta t) + \mathbf{R}_i(\mathbf{x}, t)\right) \\ & + O(\Delta t^3). \end{aligned} \tag{30}$$

Since $\mathbf{R}_i(\cdot, t + \Delta t)$ depends on $\mathbf{g}_i(\cdot, t + \Delta t)$, both through $\mathbf{M}$ and through the non-equilibrium part involving $\nabla\mathbf{M}$, this system is implicit.

The usual procedure[22] defines new variables

$$\overline{\mathbf{g}}_i(\mathbf{x}', t') = \mathbf{g}_i(\mathbf{x}', t') - \frac{\Delta t}{2}\mathbf{R}_i(\mathbf{x}', t'), \tag{31}$$

for which the previously implicit system (30) becomes fully explicit,

$$\overline{\mathbf{g}}_i(\mathbf{x}+\boldsymbol{\xi}_i\Delta t, t+\Delta t)-\overline{\mathbf{g}}_i(\mathbf{x},t)=-\frac{\tau_{\mathrm{D}}\Delta t}{\tau_{\mathrm{D}}+\Delta t/2}\,\mathbf{R}_i(\mathbf{x},t)+O(\Delta t^3). \tag{32}$$

However, reconstructing $\mathbf{M}$ from $\overline{\mathbf{g}}_i$ by summing (31) leads to

$$\sum_i \overline{\mathbf{g}}_i=\mathbf{M}+\frac{\Delta t}{2\tau_{\mathrm{M}}}\left(\mathbf{M}-\mathbf{M}^{(0)}\right)-\frac{\Delta t}{4}\,(u_k\nabla M_k-\mathbf{u}\nabla\cdot\mathbf{M})-\frac{\Delta t}{8\zeta}(\mathbf{M}\times\mathbf{H})\times\mathbf{M}, \tag{33}$$

where the magnetisation gradient $\nabla\mathbf{M}=-(\boldsymbol{\Lambda}-\boldsymbol{\Lambda}^{(0)}(\mathbf{M}))/(\theta\tau_{\mathrm{D}})$ depends linearly on $\mathbf{M}$ through the definition of $\boldsymbol{\Lambda}^{(0)}$. The field $\mathbf{H}$ and equilibrium $\mathbf{M}^{(0)}$ are further coupled to $\mathbf{M}$ through Maxwell's equations, $\mathbf{M}^{(0)}=\chi\mathbf{H}=\chi(\mathbf{B}+4\pi\mathbf{M})$, and χ may depend on $\mathbf{H}$ and hence $\mathbf{M}$ as well. Solving the system (33) at each lattice point for $\mathbf{M}$ thus typically requires Newton's method.

6. THE MAGNETOVISCOUS EFFECT IN POISEUILLE FLOW

Experiments[23] with Poiseuille flow of ferrofluids found that the flow rate reduced in the presence of a steady magnetic field, an effect explained theoretically[3,24] as the $\mathbf{M}\times\mathbf{B}$ torque resisting the necessary rotation of fluid parcels in this vortical flow. We assume that all variables are functions of the streamwise coordinate x only, and the fluid velocity is purely in the y direction, $\mathbf{u}=v(x)\hat{\mathbf{y}}$. In this geometry the magnetostatic form of Maxwell's equations are readily solved as

$$H_x=B_x-4\pi M_x=H_x^{(e)}-4\pi M_x,\qquad H_y=H_y^{(e)}, \tag{34}$$

where $\mathbf{H}^{(e)}=\mathbf{B}^{(e)}$ is the imposed external field. Neglecting spin viscous effects, the steady solution for the magnetisation $\mathbf{M}$ is

$$M_x=\chi\frac{H_x^{(e)}-\frac{1}{2}\Omega\tau_{\mathrm{m}}H_y^{(e)}}{1+4\pi\chi+\frac{1}{4}\Omega^2\tau_{\mathrm{m}}^2},\qquad M_y=\chi\frac{(1+4\pi\chi)H_y^{(e)}+\frac{1}{2}\Omega\tau_{\mathrm{m}}H_x^{(e)}}{1+4\pi\chi+\frac{1}{4}\Omega^2\tau_{\mathrm{m}}^2}, \tag{35}$$

where $\Omega=\partial v/\partial x$ is the vorticity. Substituting into the streamwise momentum equation, and dropping terms involving $\Omega^2\tau_{\mathrm{m}}^2$, we recover Poiseuille flow with an increased effective fluid viscosity

$$\nu_{\mathrm{eff}}=\nu+\frac{1}{4}\chi\tau_{\mathrm{m}}H_y^{(e)2}+\frac{1}{4}\chi\tau_{\mathrm{m}}H_x^{(e)2}/(1+4\pi\chi)^2. \tag{36}$$

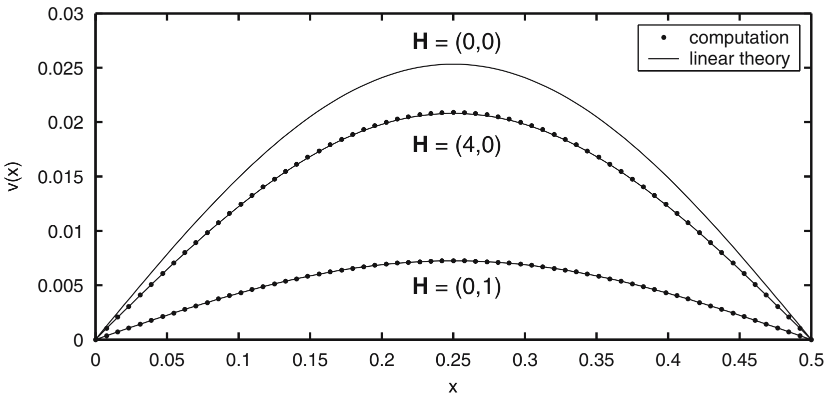

Fig. 3. Periodic channel flow computations illustrating the magnetoviscous effect for imposed fields $\mathbf{H} = (4, 0)$ and $\mathbf{H} = (0, 1)$. Only half the channel is shown.

This is a nonequilibrium, or finite τ_m, effect that cannot be captured by equilibrium descriptions of ferrofluids.[1,3,24]

Figure 3 shows results from two numerical experiments. They were conducted in a periodic channel, with a sinusoidal body force $F\sin(2\pi x)\hat{\mathbf{y}}$, to avoid issues with diffusive boundary layers on the magnetisation at the walls. The parameters were $\tau_m = 1$, $\chi = 1$ (assumed constant), $F = 0.1$, $\nu = 0.1$ in suitable dimensionless units, and a very small magnetic diffusivity $\eta = 1/1600$ that had no visible effect on the solutions. The corresponding effective viscosities from (36) are $\nu_{\text{eff}} = 0.1217$ for $\mathbf{H}^{(e)} = 4\hat{\mathbf{x}}$, and $\nu_{\text{eff}} = 0.35$ for $\mathbf{H}^{(e)} = \hat{\mathbf{y}}$. These values were used for the theoretical parabola in the figure, and are in good agreement with the numerical results.

7. CONCLUSION

The vector-valued distribution function approach from MHD may be extended to simulate the magnetisation equation (21) arising in ferrofluids, even though in general the equation takes a non-conservation form involving the fluid vorticity $\nabla \times \mathbf{u}$. This makes a lattice Boltzmann implementation much more involved than for magnetohydrodynamics,[8] or for ferrofluids in the weak magnetisation limit,[11] where the $\nabla\mathbf{M}$ terms are dropped. However, a "pure" or finite difference-free implementation is still possible using the nonequilibrium magnetic distribution function $\mathbf{\Lambda}^{(1)}$ to obtain the gradient $\nabla\mathbf{M}$.

The change of variables (31) resulting from integrating forcing terms along characteristics becomes very clumsy when the forcing depends upon the variable being forced, especially nonlinearly. Alternatives based on operator splitting for the time integration, as used by Salmon[25] for the Coriolis force, are simpler, but gave much larger errors for a given time-step. Another possibility would be to use a predictor–corrector method to solve the implicit set of equations (29) arising from the trapezium rule, without introducing the $\overline{\mathbf{g}}_i$ variables. This approach was used in Salmon's first treatment of the Coriolis force,[26] and continues to be used in liquid crystals.[14]

For general geometries and $O(1)$ magnetisation, the magnetostatic form of Maxwell's equations (7) must be solved in parallel with the fluid and magnetisation equations, although $\mathbf{H}$ is just the imposed field $\mathbf{B}^{(e)}$ in the weak magnetisation limit. In the geometry of Poiseuille flow the solution of Maxwell's equations may be written down (34), allowing a simple computation of the magnetoviscous effect.[3,23,24]

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